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TRANSVERSE-LONGITUDINAL WAVES EXCITED BY A PLECTRUM IN A PLUCKED INSTRUMENT STRING†

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A solution of the problem of the propagation of longitudinal-transverse waves in the string of a plucked instrument excited by a wedge-shaped plectrum when the plectrum has an arbitrary law of motion is obtained within the framework of the formulation proposed previously in [1]. The presence of an additional spectrum of transverse-longitudinal vibrations while the plectrum is acting is established. For the case of a wedge-shaped plectrum, moving with constant velocity, an exact non-linear solution of the problem is obtained, which confirms the possibility of using linearized equations [1, 2] to describe the free play on plucked instruments. The presence of a velocity field in the string when the action of the performer is completed, leads to a new formulation of the problem of the longitudinal-transverse vibrations of the string of a plucked instrument after the performer has completed his action, which differs from the existing formulation of the problem [3, 4]. © 2003 Elsevier Ltd. All rights reserved.

1. THE ACTION OF A HALF-PLANE ON AN UNLIMITED STRING

We will consider the action on an unlimited string of a wedge-shaped plectrum of aperture angle 2α, moving with constant velocity. The OX axis coincides with the initial position of the string. The Oy axis is directed along the normal to the surface of the plectrum. Suppose V_0 is the component of the velocity of the plectrum along the Oy axis. We will assume initially that the axis of symmetry of the plectrum is perpendicular to the Ox axis, while the component of the velocity $\tilde{V}_1$ in the plane of the plectrum is directed along this axis. The string will interact with the right edge of the plectrum at the point A, having a component of the velocity $V_1 = \tilde{V}_1 \operatorname{tg} \alpha$ along the Ox axis (Fig. 1). As long as the waves in the string from the other edge have no effect, the required solution is identical with the solution of the problem of the action of a half-plane (with the same components of the velocities), the edge of which moves at an angle $\pi/2-\alpha$ to the Ox axis.

The equations of conservation of mass and the change in the momentum for an element of a flexible string have the form

$$\rho_0 = \rho(1+e), \quad \rho_0 \mathbf{l}_{tt} = \mathbf{T}_s$$
$$e = \sqrt{(1+x_s)^2+(y_s)^2}-1 \tag{1.1}$$

where ρ_0 and ρ are the initial and present density of the string respectively, $\mathbf{T}$ is the stress vector, s is the Lagrangian coordinate, which coincides with the Cartesian coordinate of the string in the underformed state, $\mathbf{l}$ is the displacement vector of the string, having projections $s(s, t)$ and $y(s, t)$, and e is the deformation.

The longitudinal displacement and the deformation can be represented in the form [3]

$$x = \hat{x} + x_0(s), \quad x_0(s) = e_0 s, \quad e = e_0 + \hat{e}$$

where x_0 e_0 are their values corresponding to the initial stress T_0.

The second equation of (1.1) can be represented in terms of the projections onto the coordinate axes

$$\rho_0 x_{tt} = (T\cos\varphi)_s, \quad \rho_0 y_{tt} = (T\sin\varphi)_s$$
$$\cos\varphi = (1+x_s)(1+e)^{-1}, \quad \sin\varphi = y_s(1+e)^{-1} \tag{1.2}$$

where φ is the angle of deflection of an element of the string from its initial direction.

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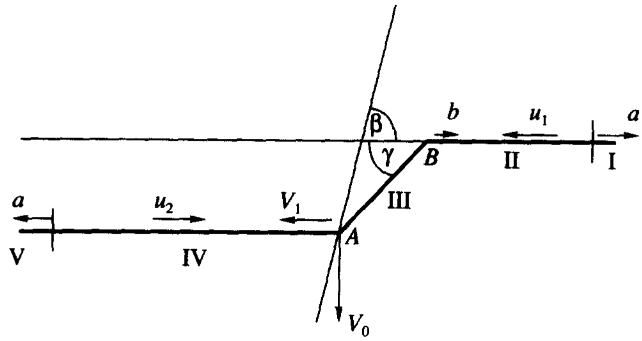

Fig. 1

We will consider the elastic case when $T = Ee$ (E is Young's modulus). It follows from the method of dimensions and similitude [1–3], that the solutions for the x and y components can be written as follows

$$y = atf_1(z), \quad x = atf_2(z), \quad z = s/at, \quad a = \sqrt{E/\rho_0}$$

The solutions for f_1 and f_2 have the simplest form: $\dot{f}_1 = \text{const}$, $\dot{f}_2 = \text{const}$, which denotes the presence of only regions of constant parameters.

As a result, a wave pattern arises, shown in Fig. 1, where I is the region ($s > at$), where waves should not occur and $y \equiv 0, x \equiv e_0 s$, II is the region ($bt < s < at$) of longitudinal motions, where $y \equiv 0$, III is the region ($-V_t < s < bt$) of longitudinal-transverse motions of the particles of the string, IV is the region ($-at < s < -V_1 t$) of longitudinal-transverse motions of the string on the surface of the plectrum, where $y_t = V_0$ (an absolutely inelastic stroke of the plectrum on the string is assumed), and V is the region ($s < -at$), where there are no longitudinal motions and $y_t = V_0$.

The condition of continuity of the flow of particles of the string, passing through the edge of the plectrum (point A), is

$$\rho_1 V_r = \rho_2 V_\tau \tag{1.3}$$

Where V_r and V_τ are the velocities of the particles along the string to the left and to the right of the point A. Since $V_r = u_2 + V_1$, this equation can be written in the form

$$\rho_1(u_2 + V_1) = \rho_2 V_\tau; \quad (1 + e_1)(u_2 + V_1) = (1 + e_2)V_\tau \tag{1.4}$$

where u_2 is the longitudinal component of the velocity of the particles of the string in region IV, ρ_1 and e_1 are the density and deformation in region IV, and ρ_2 and e_2 are the same quantities in region III. The projections of the velocity of the particles of the string onto the section AB have the form

$$V_y = V_1' \sin\beta = V_0 - V_\tau \sin\gamma, \quad V_x = V_1' \cos\beta = V_1 - V_\tau \cos\gamma \tag{1.5}$$

where V_1' and β are the modulus of the velocity and the direction of motion of the particles of the string on this path, and γ is the angle of deflection of the string at the point B.

We will consider the case when there is no friction between the string and the half-plane at the point A. Continuity of the deformations $e_1 = e_2$ then occurs. It follows from condition (1.3) that in this case

$$u_2 + V_1 = V_\tau$$

Relations similar to the relations obtained previously in [1] hold on the transverse wave B, which moves with velocity b. These relations express the law of change of momentum

$$\rho_0(b + u_1)(V_1' \cos\beta - u_1) = T(\cos\gamma - 1)(1 + e_1) \tag{1.6}$$

$$\rho_0(b + u_1)V_1' \cos\beta = T \sin\gamma(1 + e_1) \tag{1.7}$$

and the geometrical relations

$$b \sin\gamma = V_1' \sin(\beta - \gamma), \quad \text{tg}\gamma = V_0(b + V_1)^{-1} \tag{1.8}$$

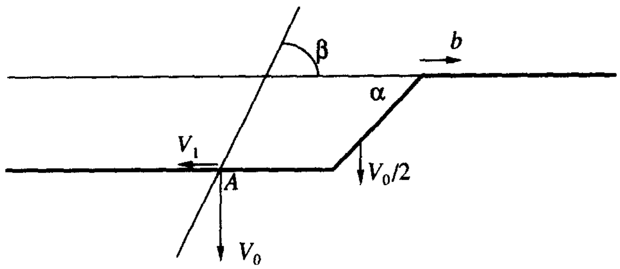

Fig. 2

where u_1 is the velocity of the particles of the string in region II.

Subtracting Eq. (1.7), multiplied by cos γ, from Eq. (1.6), multiplied by sin γ, and taking relation (1.8) into account, we obtain that

$$\rho_0(b+u_1)^2 = T(1+e_1) = E(e_0+\hat{e})(1+e_0+\hat{e}), \quad \hat{e} = e_1-e_0 \tag{1.9}$$

whence it follows that

$$b+u_1 = a\sqrt{(1+e_0+\hat{e})(e_0+e)}$$

Since the condition $u_1 = a\hat{e}$ is satisfied, we have

$$b = a(\sqrt{(e_0+\hat{e})(1+e_0+\hat{e})}-\hat{e}) \tag{1.10}$$

Multiplying Eq. (1.7) by $b + u_1$ and taking relation (1.10) into account, we obtain

$$V_y = V_1'\sin\beta = (b+u_1)\sin\gamma, \quad V_y = a[\sqrt{(e_0+\hat{e})(1+e_0+\hat{e})}]\sin\gamma \tag{1.11}$$

Taking expression (1.9) into account, we conclude from Eq. (1.6) that

$$u_1(1+\cos\gamma) = V_1(1-\cos\gamma)+(b+u_1)(1-\cos\gamma)$$

Hence, we obtain the relation

$$u_1 = (b+u_1+V_1)\mathrm{tg}^2(\gamma/2) \tag{1.12}$$

and also

$$\hat{e} = (\sqrt{(e_0\mp e)(1+e_0\mp e)}+V_1/a)\mathrm{tg}^2(\gamma/2) \tag{1.13}$$

Thus, the quantities γ, $\hat{e}$, V_x, V_e, $u_1 = a\hat{e}$ are found from Eqs (1.5), (1.10), (1.11), and (1.13) in terms of the velocity of motion of the plectrum and the initial tension e_0. When $\alpha = 0$ the results are identical with those obtained previously in [2].

These results enable us to solve the problem of the excitation, by a wedge-shaped plectrum with an aperture angle 2α, of a string of finite dimensions. Only regions of constant parameters will arise in this problem: (1) when longitudinal waves are reflected from the middle of the plectrum with the occurrence behind them of regions at rest, and (2) when a longitudinal wave, reflected from the middle of the plectrum, passes through the point A, leading to the occurrence of two longitudinal waves, one of which begins to move backwards, to the middle of the plectrum, and the second of which propagates along the section AB. Behind this second wave there will be a region of constant parameters, the changing tension in which gives rise to the occurrence of a new transverse wave, propagating from the point A and leading to a change in the angle of inclination of the string along this section.

When the new longitudinal wave reaches the point B a configuration arises [1]. consisting of two transverse waves moving in different directions with respect to the region which has already experienced the action of the newly occurring longitudinal waves. Similar situations will also occur when the longitudinal wave reflected from the clamp encounters the transverse wave moving towards the clamp.

The solution can be extended to the case when the axis of symmetry of the plectrum is not perpendicular to the *Ox* axis, if we use different values of V_1 for the components of the velocities of points of deflection of the string on different sides of the plectrum.

It should be borne in mind that the formulae of Section 1 hold when $V_1 < b$. If $V_1 > b$, the configuration shown in Fig. 2 occurs. The edge of the plectrum overtakes the transverse wave. It is easy to show that in this case the following relations hold in region III

$$V_y = V_0/2, \quad u_1 = u_2 = 0$$

2. THE INTERACTION OF THE PLECTRUM WITH THE STRING

The linear approximation. From formulae (1.6), (1.8), (1.11) and (1.13), taking into account the fact that quantities γ, $\hat{e}, \bar{V}_0 = V_0/a$ are small, we successively obtain

$$\begin{aligned}
\gamma &= 2(\sqrt{e_0(1+e_0)} + V_1/a)^{-1/2}\hat{e}^{1/2} + O(\hat{e}^{3/2}), \\
\bar{V}_0 &= 2(e_0(1+e_0) + V_1/a)^{1/2}\hat{e}^{1/2} + O(\hat{e}^{3/2}) \\
\bar{V}_y &= 2\sqrt{e_0(1+e_0)}(\sqrt{e_0(1+e_0)} + V_1/a)^{-1/2}\hat{e}^{1/2} + O(\hat{e}^{3/2}), \quad \bar{V}_x = \hat{e} + O(\hat{e}^2)
\end{aligned} \tag{2.1}$$

Since $\bar{V}_1 \sim V_0$, we have $V_1 \sim V_0$, with the exception of the cases $\alpha \to \pi/2$, when the velocity V_1 may be of the order of the velocity of the transverse wave b. If we exclude the case $\alpha \to \pi/2$, formulae (2.1) take the form

$$\begin{aligned}
\gamma &= 2[e_0(1+e_0)]^{-1/4}\hat{e}^{1/2} + O(\hat{e}^{3/2}), \quad \bar{V}_0 = 2[e_0(1+e_0)]^{1/4}\hat{e}^{1/4} + O(\hat{e}^{3/2}), \\
\bar{V}_y &= 2[e_0(1+e_0)]^{1/4}\hat{e}^{1/2} + O(\hat{e}^{3/2}), \quad \bar{V}_x = \hat{e} + O(\hat{e}^2)
\end{aligned} \tag{2.2}$$

Similar expansions

$$Y = \hat{e}^{1/2}Y_1 + \hat{e}^{3/2}Y_2 \ldots, \quad \bar{X} = X - e_0 s = \hat{e}X_1 + \hat{e}^2 X_2 \ldots$$

were used in [3] to obtain, from Eq. (1.1), the linearized equations of the first approximation

$$Y_{1tt} = b^2 Y_{1ss} \tag{2.3}$$

$$X_{1tt} = a^2\left[X_{1s} + \frac{1}{2(1+e_0)^2}(Y_{1s})^2\right]_s \tag{2.4}$$

and of the second approximation

$$\begin{aligned}
Y_{2tt} &= b^2 Y_{2ss} + a^2\left[\frac{1}{(1+e_0)^2}X_{1s}Y_{1s} + \frac{1}{2(1+e_0)^3}(Y_{1s})^3\right]_s \\
X_{2tt} &= a^2\left[X_{2s} + \frac{1}{(1+e_0)^2}Y_{1s}Y_{2s} - \frac{1}{(1+e_0)^3}X_{1s}(Y_{1s})^2 - \frac{3}{8(1+e_0)^4}(Y_{1s})^4\right]_s
\end{aligned}$$

We will solve Eqs (2.3) and (2.4) for the case when a half-plane, moving with a variable velocity $V_0(t)$, acts on the string. We will then extend this result to the interaction of a triangular plectrum with a bounded string (rigidly clamped at the ends), and then to the case of a plectrum of arbitrary shape moving with an arbitrary velocity.

The solution of Eq. (2.3) in this case will be

$$Y_1(s,t) = \begin{cases} \hat{e}^{-1/2}y_0(t), & s < s^*(t) \\ \hat{e}^{-1/2}y_0[\tilde{t}(t - s/b)], & s^*(t) < s < bt, \quad y_0(t) = \int\limits_0^t V_0(t')dt' \\ 0, & s > bt \end{cases}$$

where $s^*(t)$ is the coordinate of the point A and $\tilde{t}(z)$ is the solution of the equation $\tilde{t}-s^*(t)/b=z$. For motion with a constant velocity we obtain

$$Y_1(s,t)=\begin{cases}\hat{e}^{-1/2}V_0t, & s<V_1t\\ \hat{e}^{-1/2}\dfrac{V_0b}{b+V_1}\left(t-\dfrac{s}{b}\right), & -V_1t<s<bt,\\ 0, & s>bt\end{cases}$$

These solutions are identical with those obtained previously in the non-linear formulation as $\hat{e}\to 0$.

If we take into account reflections from the middle of the plectrum and from the clamping points L_+ and L_- we obtain the following results

$$\hat{e}^{1/2}Y_l^{\pm}(s,t)=\sum_{n=0}^{\infty}\frac{V_0b}{b+V_1}\left(t\mp\frac{s-s_n^{\pm}\pm 2nL_{\pm}}{b}\right)-\sum_{n=1}^{\infty}\frac{V_0b}{b+V_1}\left(t\pm\frac{s-s_n^{\pm}\mp 2nL_{\pm}}{b}\right)$$

where s_n^+ and s_n^- are coordinates of the points of deflection at the instants when the reflected waves arrive; the superscripts + and − relate to the regions to the right and left of the plectrum respectively.

The solution of Eq. (2.4) will be sought in the form

$$\begin{aligned}&x_{II}=f_1(\xi),\quad x_{IV}=f_2(\eta),\quad x_{III}=f_3(\xi)+f_4(\eta)+F(s,t)\\&(x_I=x_V=0)\\&\xi=t-s/a,\quad \eta=t+s/a\end{aligned}\tag{2.5}$$

where $F(s,t)$ is a particular solution of Eq. (2.3). Substituting it into Eq. (2.3) we have

$$F''(s,t)=\frac{a^2}{b(1+e_0)^2(b^2-a^2)}Y'(t-s/b)Y''(t-s/b)\tag{2.6}$$

i.e. this particular solution can be represented as

$$F(s,t)=F(t-s/b)$$

The difference between the formulations of the problems for determining the longitudinal displacements X_1 and X_2 and the transverse displacements Y_1 and Y_2 is the fact that, on the transverse waves and at points of contact with the plectrum

$$s=bt,\quad s^*=-\int_0^t V_1dt'$$

the first derivatives of the functions X_1 and X_2 have discontinuities [2]

$$\begin{aligned}&[X_{1t}]=\frac{b}{2(1+e_0)}(Y_{1s})^2=\frac{1}{2b(1+e_0)}Y_0'^2\left(t-\frac{s}{b}\right)\\&[X_{1s}]=-\frac{1}{2(1+e_0)}(Y_{1s})^2=\frac{1}{2b^2(1+e_0)}Y_0'^2\left(t-\frac{s}{b}\right)\end{aligned}\quad\text{when } s=bt\tag{2.7}$$

$$[X_{1t}]=0,\quad \left[X_{1s}+\frac{1}{2(1+e_0)}(Y_{1s})^2\right]=0\quad\text{when } s^*=-\int_0^t V_1dt'\tag{2.8}$$

(the square brackets denote a discontinuity of the components of the corresponding quantities behind and in front of the wave, and also to the right and left of the point of contact of the edge of the plectrum with the string).

Using these conditions and substituting the solutions for X_1 in the form (2.5) into (2.7) and (2.8), we have, for the case of a constant velocity in the different regions,

$$x_{II} = f_1(\xi) = A\xi, \quad x_{IV} = f_2(\eta) = A\eta$$
$$x_{III} = f_3(\xi) + f_4(\eta) + F(t - s/b) = A(a-b)s/b$$
$$A = \frac{bV_0^2}{4(1+e_0)^2(b+V_1)^2}$$

To consider the longitudinal motion of the particles of the string when the plectrum moves with a variable velocity it is necessary to obtain the coordinates s^* of the point at which the string interacts with the edge of the plectrum

$$(1+e_0)s^*(t) + X(s^*(t), t) = -\int_0^t V_0(t')dt'$$

The solution of this problem is very cumbersome in the general case and will not be given here.

In view of the fact that the dimensions of the plectrum are small, the effect of corrections, related to determining s^*, only arise in the second approximation, and hence the solution for $X_1(s, t)$ in region III can be represented by the relation

$$X_1(s,t) = \frac{1}{2}\left(1-\frac{a}{b}\right)[F'(\xi) - F'(\eta)] + \frac{a}{4b^2(1+e_0)^2}\left[\int_0^t Y_0'^2(\xi)d\xi + \int_0^t Y_0'^2(\eta)d\eta\right]$$

3. ANALYSIS OF THE RESULTS

Two conclusions follow from the results obtained. The first is related to the fact that, when the performer is acting, transverse-longitudinal vibration frequencies occur, which differ from the frequencies of the fundamental tone and of the overtones of the string. The time of this action, as a rule, varies between 0.01 s and 0.05 s. (This time was, in particular, determined by the second author of this paper together with A. V. Bryukvin by measuring, using an electronic chronograph, the period of closure of an electric circuit consisting of a metal string and a metal plectrum.) After this time, along parts of the string from the plectrum to the clamp, the transverse and longitudinal waves are able to travel and be reflected many times. Hence, along these parts vibrations occur, the fundamental tone of which will be higher than the fundamental tone of the string as a whole, and which will be received by the clamp and the sounding board giving an additional spectrum of vibrations. By expanding the solutions for the longitudinal and transverse waves presented in Section 2 in Fourier series we can obtain the amplitude of these vibrations, and also take into account the frequency shift while the plectrum is being displaced and the increase in the distance from the clamp to the points where the string touches the edge of the plectrum.

Nevertheless, according the approach proposed previously [2], because the width of the plectrum is small, the spectrum of the vibrations can be determined assuming that the distance from the clamp to a fixed point of the plectrum (for example, its centre) remains unchanged, and the frequency shift and the correction to the value of the amplitude of the vibrations can be obtained by the perturbation method.

It is easy to estimate the change in the frequencies, assuming that the velocity of the plectrum is constant, from the formula.

$$\frac{\Delta\omega}{\Delta t} = \frac{\pi b \Delta l}{L^2 \Delta t} = \frac{\pi b \Delta l \tilde{V}_1}{L^2 \Delta h}$$

where Δl is the initial half-width of the plectrum at the point where it covers the string. Δh is the length of the coverage and L is the distance from the middle of the plectrum to the clamp.

The second conclusion is related to the use of the solutions from Section 2 to determine the spectrum of the transverse-longitudinal vibrations of the string after the action of the performer is completed.

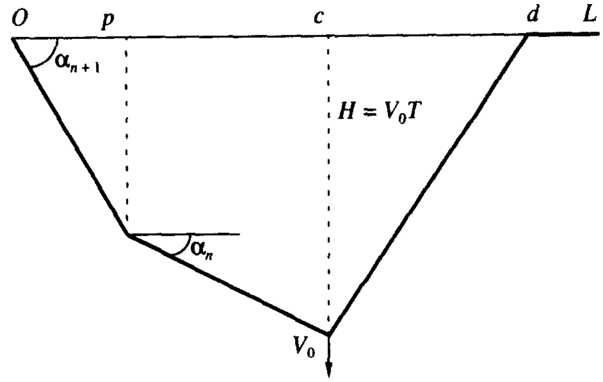

Fig. 3

The solutions obtained above enable us to find the distribution of the longitudinal and transverse components of the displacement vectors and the velocity vectors at the instant when the action of the plectrum (or the finger of the performer) is completed. Assuming the distributions obtained as the initial conditions for Eqs (2.3) and (2.4) and taking into account the boundary conditions at the clamping points, we can solve the problem of determining the spectrum of the transverse and longitudinal vibrations of a string after the action of the performer.

Up to the present time, when determining the spectra of both the transverse and transverse-longitudinal vibrations in the strings of plucked instruments using the existing formulation [4], the initial distribution of the displacements was specified (in connection with the fact that the string along the sections from the clamping to the location of the action of the performer has a rectilinear form), while the initial velocities were taken as zero. When the ends of the string are rigidly clamped the spectra of transverse vibrations [4, 5] and longitudinal vibrations [6] were derived. It is well known that the expression for the transverse displacements is the superposition of terms of the form $f_n \sin \omega_n t$, where the coefficients f_n are determined by an expansion in a Fourier series of the initial triangular distribution.

In fact, the spectrum of the vibrations (after the action of the performer is completed) will represent superpositions of terms of the form $f_n \sin \omega_n t$ and $\varphi_n \cos \omega_n t$, where the coefficients f_n and φ_n are determined by the corresponding expansion in a Fourier series of the initial (non-triangular) distribution of the transverse components of the displacements and velocities. Therefore, the spectrum of the transverse-longitudinal vibrations has the same frequencies as for the previous formulation, quantitatively and qualitatively differing from it by the value of the amplitude of the vibrations (and consequently the energies of the modes also), and hence also in having a phase shift.

In the general case, the calculations lead to lengthy expressions.

We will consider the spectrum of the transverse vibrations. We will represent the plectrum in the form of a point moving with velocity $V_0 = \text{const}$, and for clarity we will consider the case (corresponding to actual conditions of the play), when the waves are multiply reflected from the clamp closest to the point of action of the plectrum. Such cases lead to the next possible form of the string and the distribution of the transverse components of the velocities, shown in Fig. 3. (There are no transverse components of the velocity along the sections $0 \leqslant s \leqslant a$ and $d \leqslant s \leqslant L$.) The spectrum of the transverse vibrations is given by the expression

$$\begin{aligned}
\widehat{e}^{\,1/2} Y_1(s,t) &= \frac{2V_0H}{b}\sum_{n=1}^{\infty} \sin\frac{\pi n s}{L}\left[A_n \sin\frac{\pi n}{L}bt + B_n \cos\frac{\pi n}{L}bt\right] \\
A_n &= \frac{2LV_0}{\pi^2 n^2 b}\left(\cos\frac{\pi n p}{L} - \cos\frac{\pi n d}{L}\right) \\
B_n &= \frac{2V_0}{b}\left[\frac{L}{\pi^2 n^2}\left(k\sin\frac{\pi n c}{L} - \sin\frac{\pi n d}{L} + \sin\frac{\pi n p}{L}\right) + \frac{d-bt}{\pi n}\cos\frac{\pi n c}{L}\right]
\end{aligned} \tag{3.1}$$

where k is the number of reflections from the clamp and $H = V_0 t$.

The spectrum of transverse vibrations (3.1) differs from that given in the literature [4, 5] in amplitude and frequency shift.

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